ILLUSTRATION OF THE ELLIPTIC INTEGRAL OF THE FIRST KIND BY A CERTAIN LINK-WORK.

BY ARNOLD EMCH

1. THE element of the link-work consists of a cell formed by six bars OA_1, A_1B_1, B_1A_2, A_2O, QB_1, and OQ, fig. 1. The first four are of equal

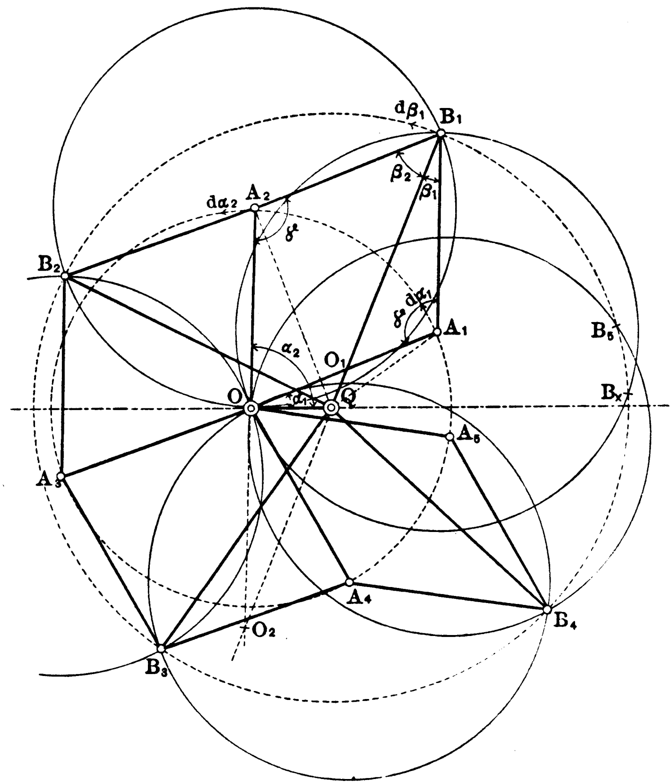

FIG. 1.

length and form a rhombus whose only fixed point is O, while QB_1 is of different length and movable about the fixed point Q. The bar OQ is fixed. As this link-work consists of 5 joints and 6 bars it is movable and has one

degree of freedom $((2 \times 5 - 3) - 6 = 1)$.* The motion is unlimited, *i. e.*, the cell can make complete revolutions, if

$$OA_1 + A_1B_1 > OQ + QB_1.$$

It is limited if

$$OA_1 + A_1B_1 < OQ + QB_1.$$

I shall consider the motion of the cell in the first case, where it can make complete revolutions.

2. Let $\delta\alpha_1$, $\delta\alpha_2$, $\delta\beta_1$ be the infinitesimal displacements of the points A_1, A_2, B_1 in a virtual displacement of the cell; α_1, α_2 the angles which the links OA_1, OA_2 include with the positive part of the axis OQ; β_1, β_2 the angles which the links A_1B_1 and B_1A_2 include with the link QB_1; O_1 and O_2 the points of intersection of the link QB_1 with the links OA_1 and OA_2 respectively; and, finally, the variable distances $\rho_1 = QA_1$ and $\rho_2 = QA_2$. The points O_1 and O_2 are evidently the virtual centres of rotation of the links A_1B_1 and B_1A_2 respectively. Hence, from fig. 1, the relations:

$$\frac{\delta\alpha_1}{\delta\beta_1} = \frac{A_1O_1}{B_1O_1}, \qquad (1) \qquad\qquad \frac{\delta\alpha_2}{\delta\beta_2} = \frac{A_2O_2}{B_1O_2}, \qquad (2)$$

from which follows

$$\frac{\delta\alpha_1}{\dfrac{A_1O_1}{B_1O_1}} = \frac{\delta\alpha_2}{\dfrac{A_2O_2}{B_1O_2}}. \qquad (3)$$

Now
$$\frac{A_1O_1}{B_1O_1} = \frac{\sin\beta_1}{\sin\gamma}, \qquad \frac{A_2O_2}{B_1O_2} = \frac{\sin\beta_2}{\sin\gamma},$$

where *angle* $OA_1B_1 =$ *angle* $OA_2B_1 = \gamma$. Consequently,

$$\frac{\delta\alpha_1}{\sin\beta_1} = \frac{\delta\alpha_2}{\sin\beta_2}. \qquad (4)$$

As there is only one degree of freedom, the angles β_1, β_2, α_1, and α_2 may be regarded as functions of the same independent variable. This differential equation assumes a more intelligible form by introducing $QA_1 = \rho_1$ and $QA_2 = \rho_2$ as variables. Putting $OA_1 = r$, $QB_1 = R$, and $OQ = e$, we have:

$$\cos\alpha_1 = \frac{r^2 + e^2 - \rho_1^2}{2re},$$

* See Cremona, *Graphic Statics*, p. 152, and F. Reuleaux, *Kinematics of Machinery* pp. 283–294.

and by differentiation

$$\sin \alpha_1 \cdot d\alpha_1 = \frac{\rho_1 \cdot d\rho_1}{re}.$$

But

$$\sin \alpha_1 = \frac{2}{re} \sqrt{s(s-r)(s-e)(s-\rho_1)},$$

where $s = \frac{r+e+\rho^1}{2}$, or

$$\sin \alpha_1 = \frac{\sqrt{-[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2]}}{2\,re},$$

hence

$$d\alpha_1 = \frac{2\rho_1 \cdot d\rho_1}{\sqrt{-[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2]}}.$$

In the triangle A_1B_1Q

$$\sin \beta_1 = \frac{\sqrt{-[\rho_1^2-(R+r)^2][\rho_1^2-(R-r)^2]}}{2\,R\,r},$$

so that

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{4Rr\rho_1 \cdot d\rho_1}{\sqrt{[\rho_1^2-(r+e)^2][\rho_1^2-(r-e)^2][\rho_1^2-(R+r)^2][\rho_1^2-(R-r)^2]}}.$$

To abbreviate let $R+r=a$, $r+e=b$, $r-e=c$, $R-r=d$, $\rho_1^2=x$, $\rho_1 \cdot d\rho_1 = \frac{1}{2}\,dx$, so that finally

$$\frac{d\alpha_1}{\sin \beta_1} = \frac{2Rr \cdot dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}. \tag{5}$$

In a similar manner, if $QA_2=\rho_2$, and $\rho_2^2=y$,

$$\frac{d\alpha_2}{\sin \beta_2} = \frac{2Rr \cdot dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}}. \tag{6}$$

Putting

$$\int_c^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = u, \tag{7}$$

$$\int_c^y \frac{dy}{\sqrt{(y-a)(y-b)(y-c)(y-d)}} = v^*, \tag{8}$$

* As c is the smallest real value of x or y, we assumed it as the lower limit. The largest real value of x or y, b, might also be taken as the lower limit.

according to equation (4), we have:

$$v - u = h \text{ (constant)}. \tag{9}$$

By inversion of the elliptic integrals (7) and (8) the elliptic functions

$$x = \lambda(u) \quad , \; y = \lambda(v) \tag{10}$$

are obtained. In this manner the cosines of the angles a_1 and a_2 may be rationally expressed by elliptic functions, and it is found that *the difference of the arguments belonging to these angles is constant and independent of the position of the cell.*

3. As indicated in fig. 1, other equal cells $(OA_2B_2A_3 \cdot B_2Q)$, $(OA_3B_3A_4 \cdot B_3Q)$ . . . may be added to the first, which together form a general link-work. In this process of adding cells two principal cases may occur: (1) the link-work will close after a certain number of additions of cells, *i. e.*, the last point A obtained in the construction will coincide with the first of the points A; (2) the link-work does not close.

To discuss the conditions of a closed link-work assume that there are n cells in it, so that the point A_{n+1} of the nth cell $OA_nB_nA_{n+1} \cdot QB_n$ will coincide with the first point A_1. The argument belonging to the angle a_1 or the point A_1 being u, the argument of A_2 will be $u+h$, of A_3 $u+2h$, . . ., of A_{n+1} $u+nh$. But A_{n+1} coincides with A_1, hence, designating the periods of the elliptic function $\lambda(u)$ by w_1 and w_2,

$$u + nh \equiv u \pmod{w_1, w_2}.$$

This condition is satisfied if

$$h \equiv 0 \quad \left(\operatorname{mod} \frac{w_1}{n}, \frac{w_2}{n}\right),$$

or

$$h = \frac{m_1 w_1 + m_2 w_2}{n}, \tag{11}$$

where m_1 and m_2 designate integers. Consequently the problem of a closed link-work is solved if h is given one of the values contained in (11). This condition necessarily requires a special arrangement of the link-work; but it does not assign any particular value to the argument u. Thus, the first point A_1 of the link-work may be chosen anywhere on the circle having O as a centre and OA_1 as a radius; the link-work closes every time and contains n cells. This result may be stated in the theorem:

If a link-work of the prescribed kind based upon two fixed circles (one having O as a centre and r as a radius, the other Q as a centre and R as a radius) closes and contains n cells, every other link-work, based upon the same two circles, closes and contains n cells.

4. In order to reduce the integral

$$\int \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$

to Legendre's normal form, we have to notice that in the case of an unlimited motion $2r > R + e$, or $(r-e) > (R-r)$, or $(r-e)^2 > (R-r)^2$. But we have also $R + r > r + e$, and $r + e > r - e$, hence $R + r > r + e > r - e > R - r$, or

$$a > b > c > d. \tag{12}$$

In our case we always have $b > x > c$, so that according to a well-known formula*

$$\int_c^x \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} = \frac{2}{\sqrt{(a-c)(b-d)}} \mathrm{sn}^{-1} \sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}}, \tag{13}$$

with the modulus $\kappa^2 = k = \dfrac{(b-c)(a-d)}{(a-c)(b-d)}$.

Putting this integral, as in formula (5), equal to u, we have:

$$\sqrt{\frac{(b-d)(x-c)}{(b-c)(x-d)}} = \mathrm{sn}\left(\frac{\sqrt{(a-c)(b-d)}}{2} \cdot u\right), \tag{14}$$

or, putting $\dfrac{\sqrt{(a-c)(b-d)}}{2} \cdot u = w$,

$$\frac{(x-c)(b-d)}{(x-d)(b-c)} = \mathrm{sn}^2 w. \tag{15}$$

From this

$$x = \frac{c(b-d) - d(b-c)\,\mathrm{sn}^2 w}{(b-d) - (b-c)\,\mathrm{sn}^2 w}. \tag{16}$$

For $u = 0$, $x = c = (r-e)^2$. The corresponding value of y is easily found as

$$y = re + \frac{r^3 - eR^2}{r-e} = p. \tag{17}$$

* See Greenhill, *Elliptic Functions*, pp. 53-55.

This value of y belongs to the argument $v=h$, since $v-u=h$; hence the constant h is determined by

$$re+\frac{r^3-eR^2}{r-e}=\frac{c(b-d)-d(b-c)\ \mathrm{sn}^2\left(\frac{\sqrt{(a-c)(b-d)}}{2}\cdot a\right)}{b-d\ -\ (b-c)\ \mathrm{sn}^2\left(\frac{\sqrt{(a-c)(b-d)}}{2}\cdot a\right)}. \tag{18}$$

Designating the real half-period of sn w by $2K$, we have:

$$\mathrm{sn}\ (w+2K)=-\,\mathrm{sn}\,w,$$

or

$$\mathrm{sn}^2\ (w+2K)=\mathrm{sn}^2\,w,$$

i. e., $2K$ is the real period of $\mathrm{sn}^2 w$. For $w=0$, $\mathrm{sn}^2 w=0$ and $x=(r-e)^2$. For $w=K$, $\mathrm{sn}^2 w=1$ and $x=b=(r+e)^2$. For $w=2K$, $\mathrm{sn}^2 w=0$ and $x=c=(r-e)^2$. To find the corresponding value of x, belonging to $w=K/2$, we make use of the formula:*

$$\mathrm{sn}\,\frac{K}{2}=\frac{1}{\sqrt{1+\sqrt{k'}}}, \tag{19}$$

where

$$k'=\frac{(a-b)(c-d)}{(a-c)(b-d)}$$

is the complementary modulus. Thus, for $w=K/2$, from formula (16) we obtain

$$x=\frac{b(c-d)+c(b-d)\sqrt{k'}}{(c-d)\ +\ (b-d)\sqrt{k'}}. \tag{20}$$

5. Example of 3 Cells. As the period of $\mathrm{sn}^2 w$ is $2K$, we have to put $w=2K$, in order to obtain the relation of R, r, e, in this particular closed link-work. Designating $\mathrm{sn}\,w/3$ simply by S, we have:

$$\mathrm{sn}\ w=\frac{3S-4(1+k)S^3+6kS^5-k^2S^9}{1\ -6kS^4+4(1+k)kS^6-3k^2S^8}, \tag{21}$$

and since $\mathrm{sn}\,2K=0$, the condition becomes

$$k^2S^8-6kS^4+4(1+k)\ S^2-3=0. \tag{22}$$

* For the formulas used and developed here and in the next two sections we refer to Greenhill, *loc. cit.*, pp. 120-121.

According to formulas (17) and (15):

$$S^2 = \frac{(x-c)(b-d)}{(x-d)(b-c)}.$$

Designating this expression by q, the required condition is

$$k^2q^4 - 6\,kq^2 + 4\,(1+k)\,q - 3 = 0. \tag{23}$$

Substituting in this expression the values of k and q in terms of R, r, e, it is easily found that condition (23) reduces to

$$R = r. \tag{24}$$

Thus, the three cell link-work is completely determined by fixing r and c. In fig. 2, $OQ = e$ and $OA_1 = r$. Now $R = r$, hence, in this case, $A_1B_1 = QB_1$. Having fixed the point B_1 it is an easy matter to complete the construction of

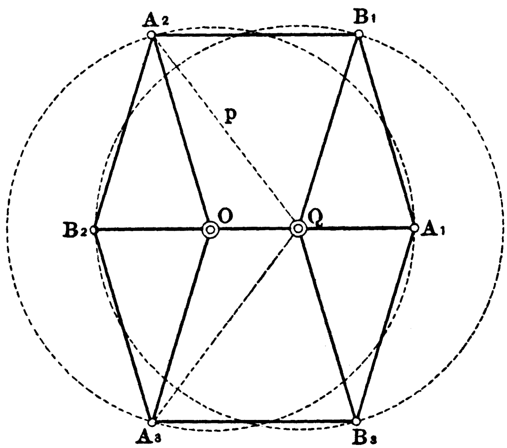

FIG. 2.

the cells $A_1B_1A_2O$, $OA_2B_2A_3$, $OA_3B_4A_1$. It is seen that $QB_1 = QB_2 = QB_4$, so that also $QB_2A_2B_1$, $QB_1A_1B_4$, $QB_4A_3B_2$ may be considered as cells of the link-work. The points O, A_1, A_2, A_3 may be interchanged with the points Q, B_1, B_2, B_3 without changing the character of the link-work.

6. Example of 4 Cells. In this case the value of y as given by formula (17) is also equal to the value of x in formula (20), *i. e.*,

$$p = \frac{b(c-d) + c(b-d)\sqrt{k'}}{(c-d) + (b-d)\sqrt{k'}}.$$

Substituting in this equation for a, b, c, d, p, and k' their values in terms of R, r, e, the condition between R, r, and e is found:

$$2r^2 = R^2 + e^2. \tag{25}$$

In this case the value of $x = p$ is

$$p = r^2 - e^2 = R - r^2,$$

as is also seen from fig. 3, in which $x = \overline{QA_2}^2 = \overline{OA_2}^2 - \overline{OQ}^2 = r^2 - e^2$, and also $x = \overline{QB_2}^2 - \overline{A_2B_2}^2 = R^2 - r^2$. From this figure it is apparent that during the motion the following groups of parallel links are maintained:

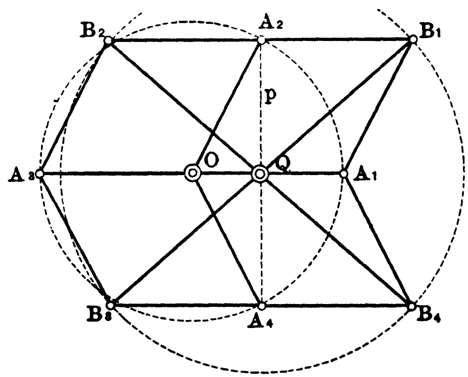

FIG. 3.

$$A_1B_1 \parallel OA_2 \parallel A_3B_2\,, \qquad A_1B_4 \parallel OA_4 \parallel A_3B_3\,,$$
$$B_1A_2 \parallel OA_1 \parallel B_4A_4\,, \qquad B_2A_2 \parallel OA_3 \parallel B_3A_4\,.$$

It follows from this that during the motion

$$B_1B_4 = A_2A_4 = BB_3,$$

and

$$B_1B_2 = A_1A_3 = B_4B_3.$$

Consequently, the points $B_1B_2B_3B_4$ always form a parallelogram, in which

$$QB_1 = QB_2 = QB_3 = QB_4.$$

But

$$B_1B_3 = QB_1 + QB_3\,,$$

and

$$B_2B_4 = QB_2 + QB_4\,,$$

hence

$$B_1B_3 = B_2B_4.$$

The parallelogram has, therefore, equal diagonals, and is a rectangle. The

closed link-work is, consequently, also completely determined by connecting the points B_1 and B_3, and B_2 and B_4 by links of equal length, and assuming

$$2r = \sqrt{2(R^2+e^2)} > R+e \,,$$

where e is any real quantity satisfying the implied condition.

These two links always cross each other at a point Q which does not change its distance from O during the motion.

7. The Open Link-Work. Consider a link-work of the prescribed kind which does not close or which is not completed so as to form a closed link-work. Suppose there are m cells in the link-work, and that the last cell does not overlap the first.* In this manner an angle $A_{m+1}OA_1$ is formed between the last and first cell. This angle, which will be designated by ϕ, is variable during the motion, and can be expressed by elliptic functions, for,

$$\phi = a_{m+1} - a_1 \tag{26}$$

is a function of the argument u.

The condition for a maximum or minimum of the angle ϕ is

$$\frac{d\phi}{du} = \frac{da_{m+1}}{dx_{m+1}} \cdot \frac{dx_{m+1}}{du} - \frac{da_1}{du_1} \cdot \frac{dx_1}{du} = 0 \,. \tag{27}$$

According to previous formulas

$$\frac{da}{dx} = \frac{1}{\sqrt{-(x-b)(x-c)}} \,, \text{ and } \frac{dx}{du} = \sqrt{(x-a)\,(x-b)\,(x-c)\,(x-d)}.$$

Substituting these expressions, with the proper indices, in (27), the condition reduces to

$$(x_1-a)(x_1-d) = (x_{m+1}-a)(x_{m+1}-d) \,,$$

or

$$x_1^2 - x_{m+1}^2 = (a+d)(x_1 - x_{m+1}) \,. \tag{28}$$

This equation is satisfied in two ways:

(1) when $\quad x_1 = x_{m+1}$, (29)

(2) when $\quad x_1 + x_{m+1} = a + d = 2\,(R^2 + r^2)$. (30)

In the first case the condition $x_1 = x_{m+1}$ does not assign any relation between R, r, and e and holds therefore for every proper link-work.

Considering a complete revolution of a link-work, fig. 1, it can easily be proved that there are only two positions of the link-work possible where

* This assumption is made in order to have a clearer idea of the link-work, although the results hold also in the most general case.

$x_1 = x_{m+1}$. This is the case every time that the cell has a symmetrical position with regard to the axis OQ, which, in these cases, bisects the open space of the link-work. Suppose now that the link-work makes a complete revolution, starting from the position of the maximum angle. The angle cannot pass through zero, because the system would then be permanently closed, so that there must be a minimum between the two maxima. Similarly there must be a maximum between two minima. This result may be summed up in the theorem:

The angle formed by an open link-work can assume only one maximum and one minimum during a complete revolution.

The maximum and minimum angles are both bisected by the diameter OQ.

If the angle becomes zero, it will remain zero. In this case we still have $x_1 = x_{m+1}$ (coincident); but for every position of the link-work. Thus, we see that the case of a closed link-work is included in case (1). The second condition $x_1 + x_{m+1} = 2\ (R^2 + r^2)$ can only be satisfied in a singular case, since $x_1 + x_{m+1}$, for all possible link-works, with constant values of R and r, may be considered as a function of m and e, having for all values of m and e a constant value. From formula (16) it appears that $x_1 + x_{m+1}$ can be independent of m and e only if $e = 0$. In this case $x_1 + x_{m+1} = 2r^2$, and, according to (30), $\mathrm{R} = 0$. There is no proper link-work.

Without entering into mechanical details of the link-work it is interesting to mention the seemingly paradoxical fact, that all our link-works have one degree of freedom in their motion, although the closed link-work satisfies the condition of a rigid frame-work.

8. Geometrical Transformation of the Link-Work. With $A_1, A_2, A_3, \ldots$, in the previous figures, as centres and r as a radius describe a series of circles. These circles all pass through O and intersect the circle of centre Q and radius R in the points B_1, B_x; B_1, B_2; B_2, B_3; B_3, B_4; ... respectively. In a closed link-work this series of circles closes also, so that the last point of intersection B_{n+1} will coincide with the first point B_x. This result may be stated in the following form:

If two fixed circles A and B are given, a series of circles can be drawn, whose centres $A_1, A_2, A_3, \ldots$ all lie on the circle A and which all pass through the centre O of A. The first circle A_1 of this series intersects circle B in two points B_x. The second circle A_2 passes through B_1 and intersects circle B a second time in B_2. The third circle passes through B_2 and intersects B a

second time in B_3, and so forth. In this manner a series of circles is obtained which may be divided into three different classes:

I. The series is limited, *i. e.*, the construction cannot be continued indefinitely.

II. The series closes, *i. e.*, after the construction of a certain number of circles, the last point of intersection B_{n+1} will coincide with the first B_1.

III. The series is unlimited.

According to the general theorem on the link-work it follows immediately that if the series of circles closes once, it will close in all cases, no matter where the first circle of the series is drawn. If the series does not close in one case, it never will close.

9. Poncelet's Poristic Polygons and Steiner's Circular Series. The circles of the previous series all touch a circle C of centre O and radius $2r$. Applying to this series an inversion with centre O and any radius, every

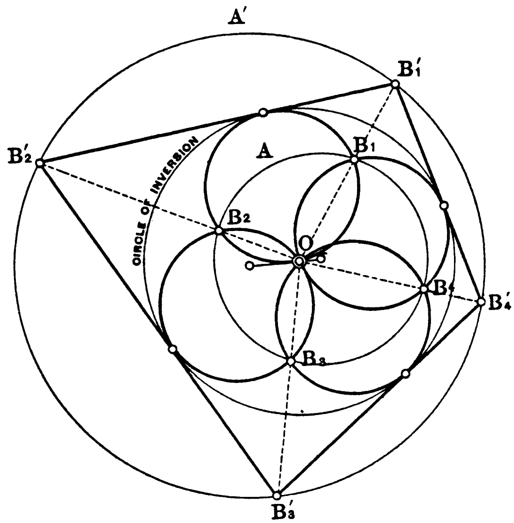

FIG. 4.

circle of the series is transformed into a straight line segment, tangent to the transformed circle of C and inscribed to the transformed circle of A. *Thus the series becomes a polygon which is inscribed to one and circumscribed to the other circle.* This is precisely the case of *Poncelet's polygons*,* fig. 4.† As

* In Poncelet's *Traité des propriétés projectives des figures* (1822) §565. See also Greenhill, *Elliptic Functions*, pp. 121–130.

† In fig. 4, C has been chosen as circle of inversion.

to the properties of closing of these polygons, it is evident that they are the same as in our link-work and the series of circles derived from it. The system of circles from which Poncelet's polygons arise may also be considered as a special case of *Steiner's circular series*,* which, in general, consists of all circles tangent to two fixed circles. From these circles a *special* series may be selected in which one point of intersection of each pair of consecutive circles always lies on a third fixed circle. These series also include the cases of Steiner's circular series where each pair of consecutive circles intersect each other under a constant angle. If this angle is zero two consecutive circles are always tangent to each other. If the first of the fixed circles of Steiner's special circular series contracts into the centre of the second fixed circle, the series arises from which Poncelet's polygons were obtained by an inversion as illustrated in fig. 4.

The properties of closing as studied by Steiner have the same character as Poncelet's polygons so that *there exists a certain equivalence between Poncelet's polygons and Steiner's circular series.*

The algebraic properties of these configurations have been studied by A. Hurwitz,† who has shown that *they rest upon the existence of more than n roots of an equation of degree n.* Their relation to the problem of the pendulum motion and Jacobi's construction for the addition theorem of elliptic functions is too well known to be repeated here. The greatest interest lies in the fact that the mechanical and geometrical interpretation of the elliptic integral of the first kind as given in this paper, leads in a simple manner to Poncelet's and Steiner's construction.

MANHATTAN, KANSAS, SEPTEMBER, 1899.

---

* Steiner's *Werke*, Vol. I, pp. 19–76 and especially pp. 43–44.

† Mathematische Annalen, Vol. 15, pp. 8–15 and Vol. 19, pp. 56–66.